Mathematics: analysis and approaches
Higher level
Paper 3

Name

worked solutions V3

Date: ____________________

1 hour

Instructions to candidates

- Write your name in the box above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

1. [Maximum mark: 29]

(a) (i) from formula booklet: $\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$

Hence, $\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = \arctan(x)\Big]_0^1 = \arctan(1)-\arctan(0) = \frac{\pi}{4}-0=\frac{\pi}{4}$ ***Q.E.D.***

(ii) $\int_0^k \frac{1}{1+x^2}\,\mathrm{d}x = \arctan(k)$

(b) (i) $\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{\mathrm{A}}{1-x}+\frac{\mathrm{B}}{1+x}$

Multiplying both sides by $(1-x)(1+x)$ and equating numerators gives:

$1=\mathrm{A}(1+x)+\mathrm{B}(1-x)$

Let $x=1$: $1=2\mathrm{A} \Rightarrow \mathrm{A}=\frac{1}{2}$ Let $x=-1$: $1=2\mathrm{B} \Rightarrow \mathrm{B}=\frac{1}{2}$

Thus, $\frac{1}{1-x^2} = \frac{\frac{1}{2}}{1-x}+\frac{\frac{1}{2}}{1+x} = \frac{1}{2(1-x)}+\frac{1}{2(1+x)}$ ***Q.E.D.***

(ii) $\int_0^{\frac{1}{2}} \frac{1}{1-x^2}\,\mathrm{d}x = \int_0^{\frac{1}{2}}\left(\frac{1}{2(1-x)}+\frac{1}{2(1+x)}\right)\mathrm{d}x = \left[-\frac{1}{2}\ln|1-x|+\frac{1}{2}\ln|1+x|\right]_0^{\frac{1}{2}}$

$=\left(-\frac{1}{2}\ln\frac{1}{2}+\frac{1}{2}\ln\frac{3}{2}\right)-\left(-\frac{1}{2}\ln 1+\frac{1}{2}\ln 1\right)=\left(\frac{1}{2}\ln 2+\frac{1}{2}\ln\frac{3}{2}\right)-(0)$

$=\frac{1}{2}\ln\left(2\cdot\frac{3}{2}\right)=\frac{1}{2}\ln 3$

(iii) $\int_0^k \frac{1}{1-x^2}\,\mathrm{d}x = \int_0^k\left(\frac{1}{2(1-x)}+\frac{1}{2(1+x)}\right)\mathrm{d}x = \left[-\frac{1}{2}\ln|1-x|+\frac{1}{2}\ln|1+x|\right]_0^k$

$=\left[\frac{1}{2}\ln\frac{1}{|1-x|}+\frac{1}{2}\ln|1+x|\right]_0^k = \left[\frac{1}{2}\ln\frac{|1+x|}{|1-x|}\right]_0^k$

$=\frac{1}{2}\left(\ln\frac{|1+k|}{|1-k|}-\ln(1)\right)=\frac{1}{2}\ln\left|\frac{1+k}{1-k}\right|$

Thus, $p(k)=1+k$ and $q(k)=1-k$

[**worked solution continued on next page**]

(c) Multiplying $\frac{x^2}{(1+x)(1+x^2)}=\frac{a}{1+x}+\frac{bx+c}{1+x^2}$ by $(1+x)(1+x^2) \Rightarrow \quad x^2=a(1+x^2)+(bx+c)(1+x)$

$x=-1:\ 1=2a \ \Rightarrow \ a=\frac{1}{2} \ \Rightarrow x^2=\frac{1}{2}(1+x^2)+(bx+c)(1+x) \ \Rightarrow \ \frac{1}{2}x^2-\frac{1}{2}=(bx+c)(1+x)$

$x=0:\ -\frac{1}{2}=c \ \Rightarrow \ \frac{1}{2}x^2-\frac{1}{2}=\left(bx-\frac{1}{2}\right)(1+x)$

$x=1:\ 0=\left(b-\frac{1}{2}\right)(2) \ \Rightarrow \ b=\frac{1}{2}$ Thus, $a=\frac{1}{2},\ b=\frac{1}{2},\ c=-\frac{1}{2}$

(d) (i) $I=\int\frac{x^2}{(1+x)(1+x^2)}\mathrm{d}x=\int\left(\frac{\frac{1}{2}}{1+x}+\frac{\frac{1}{2}x-\frac{1}{2}}{1+x^2}\right)\mathrm{d}x=\frac{1}{2}\ln|1+x|+\int\frac{\frac{1}{2}x}{1+x^2}\mathrm{d}x-\int\frac{\frac{1}{2}}{1+x^2}\mathrm{d}x$

$\int\frac{\frac{1}{2}}{1+x^2}\mathrm{d}x=\frac{1}{2}\arctan(x)$

$\int\frac{\frac{1}{2}x}{1+x^2}\mathrm{d}x \ \Rightarrow$ let $u=1+x^2 \ \Rightarrow \ \mathrm{d}u=2x\mathrm{d}x \ \Rightarrow \ \frac{1}{4}\mathrm{d}u=\frac{1}{2}x\mathrm{d}x$

$\int\frac{\frac{1}{2}x}{1+x^2}\mathrm{d}x=\frac{1}{4}\int\frac{1}{u}\mathrm{d}u=\frac{1}{4}\ln u=\frac{1}{4}\ln(1+x^2)$

Hence, $I=\int\frac{x^2}{(1+x)(1+x^2)}\mathrm{d}x=\frac{1}{2}\ln|1+x|+\frac{1}{4}\ln(1+x^2)-\frac{1}{2}\arctan(x)+C$

$$=\frac{2}{4}\ln|1+x|+\frac{1}{4}\ln(1+x^2)-\frac{1}{2}\arctan(x)+C$$

$$=\frac{1}{4}\ln\left[(1+x)^2\right]+\frac{1}{4}\ln(1+x^2)-\frac{1}{2}\arctan(x)+C$$

Thus, $I=\frac{1}{4}\ln\left[(1+x)^2(1+x^2)\right]-\frac{1}{2}\arctan(x)+C$ ***Q.E.D.***

(ii) $I=\frac{\pi}{4}$ when $x=1$: $\frac{\pi}{4}=\frac{1}{4}\ln\left[(1+1)^2(1+1^2)\right]-\frac{1}{2}\arctan(1)+C$

$$\frac{\pi}{4}=\frac{1}{4}\ln 8-\frac{1}{2}\cdot\frac{\pi}{4}+C$$

$$C=\frac{\pi}{4}+\frac{\pi}{8}-\frac{1}{4}\ln(2^3)$$

$$C=\frac{3\pi}{8}-\frac{3}{4}\ln(2) \quad \text{OR} \quad C=\frac{3\pi}{8}-\frac{1}{4}\ln(8)$$

2. [Maximum mark: 25]

(a) $u_{n+1} = \sqrt{1+u_n}$

(b) (i)

n	u_n
1	1.4142
2	1.5538
3	1.5981
4	1.6118
5	1.6161
6	1.6174
7	1.6179
8	1.6180
9	1.6180
10	1.6180

(ii) $\lim_{n\to\infty}(u_n - u_{n+1}) = 0$

(iii) As $n \to \infty$, $k(1) = u_n$. Since $\lim_{n\to\infty}(u_n - u_{n+1}) = 0$, then also $u_n = u_{n+1}$ as $n \to \infty$.

Substituting into $u_{n+1} = \sqrt{1+u_n}$, gives $k(1) = \sqrt{1+k(1)}$

(iv) $k(1) = \sqrt{1+k(1)} \Rightarrow [k(1)]^2 = 1+k(1) \Rightarrow [k(1)]^2 - k(1) - 1 = 0$ and $k(1) > 0$

$$k(1) = \frac{-(-1)+\sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} \Rightarrow k(1) = \frac{1+\sqrt{5}}{2}$$ **Q.E.D.**

(c) $k(2) = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}} > 0$

As $n \to \infty$, $k(2) = u_n = u_{n+1}$. Hence, $k(2) = \sqrt{2+k(2)}$.

$$[k(2)]^2 = 2+k(2) \Rightarrow [k(2)]^2 - k(2) - 2 = 0 \Rightarrow k(2) = \frac{-(-1)+\sqrt{(-1)^2 - 4(1)(-2)}}{2(1)}$$

Thus, $k(2) = \frac{1+\sqrt{9}}{2} = 2$

(d) $k(a) = \sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\cdots}}}}} > 0$

As $n \to \infty$, $k(a) = u_n = u_{n+1}$. Hence, $k(a) = \sqrt{a+k(2)}$.

$$[k(a)]^2 = a+k(2) \Rightarrow [k(2)]^2 - k(2) - a = 0 \Rightarrow k(a) = \frac{-(-1)+\sqrt{(-1)^2 - 4(1)(-a)}}{2(1)}$$

Thus, $k(a) = \frac{1+\sqrt{1+4a}}{2}$

(e) (i)

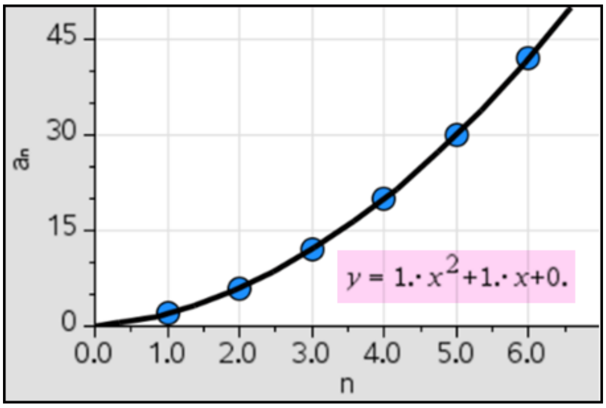
A a	B =(1+√(1+4*a[]))/2
1	1.61803398875
2	2
3	2.30277563773
4	2.56155281281
5	2.79128784748
6	3
7	3.19258240357
8	3.37228132327
9	3.54138126515
10	3.70156211872
11	3.85410196625
12	4
13	4.14005494464
14	4.27491721764
15	4.40512483795
16	4.53112887415
17	4.65331193146
18	4.77200187266
19	4.8874821937
20	5
21	5.10977222865

A a	B =(1+√(1+4*a[]))/2
22	5.21699056603
23	5.3218253805
24	5.4244289009
25	5.52493781056
26	5.62347538298
27	5.72015325446
28	5.81507290637
29	5.9083269132
30	6
31	6.09016994375
32	6.1789083458
33	6.26628129734
34	6.35234995536
35	6.43717104352
36	6.5207972894
37	6.60327780787
38	6.68465843843
39	6.76498204307
40	6.84428877023
41	6.92261628933
42	7

The first six values of a such that $k(a)$ is an integer are: $a = 2, 6, 12, 20, 30, 42$

(ii)

A n	B a_n
1	2
2	6
3	12
4	20
5	30
6	42



Quadratic regression determines that the formula for a_n is $a_n = n^2 + n$

[other methods / working possible for determining the formula]